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MATHEMATICS

ON THE CONSISTENCY AND THE POWER OF WILCOXON'S TWO SAMPLE TEST

BY

D. VAN DANTZIG

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1. WILCOXON's two sample test (1945), studied in detail by MANN and WHITNEY (1947), is based on the number $\mathbf{U}$ [1]) of "inversions", i.e. of pairs (i, j) $(i = 1, \ldots, m; j = 1, \ldots, n)$ with $\mathbf{y}_j < \mathbf{x}_i$. Here the $\mathbf{x}_i$ and the $\mathbf{y}_j$ are two independent random samples, taken from distributions with *continuous* distribution functions $F(x)$ and $G(y)$ respectively. MANN and WHITNEY determined by recursion the distribution of $\mathbf{U}$ under the hypothesis

$\mathcal{H}_0$ $\qquad F(x) = G(x) \quad \text{for all } x.$

They also proved that the distribution of $\mathbf{U}$ is asymptotically normal, and that the use of the inequality $\mathbf{U} \leqq U$, where $U = U_{a,m,n}$ is the greatest integer with

$$\beta = \mathrm{P}[\mathbf{U} \leqq U | \mathcal{H}_0] \leqq a,$$

for rejecting the hypothesis $\mathcal{H}_0$, is a consistent test for $\mathcal{H}_0$ against all alternatives $\mathcal{H}'$ with [2])

$\mathcal{H}'$ $\qquad G(x) < F(x) \text{ for all } x.$

2. The purpose of this note is to prove that WILCOXON's test is consistent under a considerably larger class of alternative hypotheses, viz. for those for which $\mathbf{x}$ is more likely than not to be smaller than $\mathbf{y}$. In fact we shall show that MANN and WHITNEY's proof with only small alterations yields the

THEOREM. *Rejection of the hypothesis $\mathcal{H}_0$ if and only if*

$$\mathbf{U} \leqq U_{a,m,n}$$

is a consistent test of $\mathcal{H}_0$ against the class of all alternatives [2])

$\mathcal{H}$ $\qquad p = \mathrm{P}[\mathbf{y} < \mathbf{x} | \mathcal{H}] < \tfrac{1}{2},$

and, for sufficiently small a, against no other alternative [2]).

The inequality $\mathcal{H}$ is equivalent with the property that the random variable $\mathbf{x} - \mathbf{y}$ has a negative median.

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[1]) Random variables will be distinguished from numbers (e.g. from the values they take in an experiment) by printing them in bold type.

[2]) Throughout the paper all $\mathbf{x}_i$ and $\mathbf{y}_j$ are supposed to be independent, all $\mathbf{x}_i$ to have the same *continuous* distribution function $F(x)$ and all $\mathbf{y}_j$ to have the same *continuous* distribution function $G(y)$.

As a by-product we obtain lower boundaries for the power-function. The proof of the theorem may easily be adapted to the twosided test $|\boldsymbol{U} - \frac{1}{2}\, mn| \geqq \frac{1}{2}\, mn - U_{\frac{1}{2}\alpha,m,n}$, which is found to be consistent against and (for sufficiently small α) only against alternatives with $p \neq \frac{1}{2}$, i.e. with a non-vanishing median of $\boldsymbol{x} - \boldsymbol{y}$.

3. We first prove the consistency, assuming p to be $< \frac{1}{2}$.
We can write $U = U_{\alpha,m,n}$ in the form

$$U = \mu_0 - c\sigma_0,$$

where $\mu_0 = \mathcal{E}[\boldsymbol{U} \,|\, H_0] = \frac{1}{2}mn$, $\sigma_0 > 0$, and (according to Mann and Whitney, $\sigma_0^2 = \mathcal{E}[(\boldsymbol{U} - \mu_0)^2 \,|\, H_0] = \frac{1}{12}\, mn\,(m+n+1)$; c, of course, like β, depends on m, n and α. Because of the asymptotic normality of $\boldsymbol{U}$

$\lim \beta = \alpha$ [3]) and $\lim c = \xi_\alpha$, where

$$\frac{1}{\sqrt{2\pi}} \int_{\xi_\alpha}^{\infty} e^{-\frac{1}{2}x^2}\, dx = \alpha.$$

We have need of the fact only that c is *bounded* when $\frac{mn}{m+n} \to \infty$.
Putting

$$\iota(z) = \begin{cases} 1 & \text{if } z \geqq 0, \\ 0 & \text{if } z < 0, \end{cases}$$

$$\boldsymbol{x}_{ij} = \iota(\boldsymbol{x}_i - \boldsymbol{y}_j),$$

we have

$$\boldsymbol{U} = \sum_{i=1}^{m} \sum_{j=1}^{n} \boldsymbol{x}_{ij}.$$

Because of the continuity $\mathrm{P}[\boldsymbol{y} \leqq \boldsymbol{x} | H] = \mathrm{P}[\boldsymbol{y} < \boldsymbol{x} | H] = p$, hence $\mathcal{E}\, \boldsymbol{x}_{ij} = p$ and $\mu = \mathcal{E}\, \boldsymbol{U} = pmn$.
Moreover, as $\iota(z)^2 = \iota(z)$,

$$\mathcal{E}\, \boldsymbol{x}_{ij}^2 = \mathcal{E}\, \boldsymbol{x}_{ij} = p.$$

As $\boldsymbol{x}_{ij}$ and $\boldsymbol{x}_{i'j'}$ are independent if $i \neq i'$ and $j \neq j'$,

$$\mathcal{E} \boldsymbol{x}_{ij}\, \boldsymbol{x}_{i'j'} = p^2 \text{ if } i \neq i'\ j \neq j'.$$

Finally, putting

$$\gamma^2 = \sigma^2_{G(\boldsymbol{x})} = \int (G(x) - p)^2\, \mathrm{d}F(x),$$
$$\varphi^2 = \sigma^2_{F(\boldsymbol{y})} = \int (F(y) - 1 + p)^2\, \mathrm{d}G(y),$$

(using the relations $p = \int G \mathrm{d}F$ and $1 - p = \int F \mathrm{d}G$), we have for $j \neq j'$;

$$\mathcal{E}\, \boldsymbol{x}_{ij}\, \boldsymbol{x}_{ij'} = \int_{-\infty}^{+\infty} \mathrm{d}F(x_i) \int_{-\infty}^{x_i} \mathrm{d}G(y_j) \int_{-\infty}^{x_i} \mathrm{d}G(y_j') = \int G^2\, \mathrm{d}F = \gamma^2 + p^2,$$

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[3]) All limits are taken for $\frac{mn}{m+n} \to \infty$ under constant α, e.g. for $m \to \infty$, $n \to \infty$, $\frac{m}{n}$ being kept constant.

and in the same way for $i \neq i'$:

$$\mathcal{E}\, \mathbf{x}_{ij}\, \mathbf{x}_{i'j} = \int (1 - F)^2\, \mathrm{d}G = \varphi^2 + p^2.$$

We obtain now

$$\mathcal{E}\, \mathbf{U}^2 = \sum_{i=1}^{m} \sum_{i'=1}^{m} \sum_{j=1}^{n} \sum_{j'=1}^{n} \mathcal{E}\, \mathbf{x}_{ij}\, \mathbf{x}_{i'j'} =$$
$$= m\, n\, p + m\, n(n-1)\, (\gamma^2 + p^2) + m(m-1)\, n(\varphi^2 + p^2) + m(m-1)\, n(n-1)\, p^2.$$

Hence, denoting by σ^2 the variance of $\mathbf{U}$ under the hypothesis $\mathcal{H}$,

$$\sigma^2 = \mathcal{E}\, \mathbf{U}^2 - (\mathcal{E}\, \mathbf{U})^2 = m\, n\, \{(m-1)\, \varphi^2 + (n-1)\, \gamma^2 + p(1-p)\}.$$

For alternatives belonging to the class $\mathcal{H}'$ the equivalent result was already obtained by MANN and WHITNEY.

Under the hypothesis $\mathcal{H}_0$ this yields by means of $p = \frac{1}{2}$: $\gamma^2 = \varphi^2 = \frac{1}{12}$, $\sigma^2 = \sigma_0^2$.

The inequalities

$$0 \leqq \gamma^2 = \int G^2 \mathrm{d}F - p^2 \leqq \int G \mathrm{d}F - p^2 = p(1-p),$$
$$0 \leqq \varphi^2 = \int (1-F)^2 \mathrm{d}G - p^2 \leqq \int (1-F)\, \mathrm{d}G - p^2 = p(1-p)$$

show that $\dfrac{\sigma^2}{\sigma_0^2}$ is *bounded*:

$$(\mathrm{I}) \quad \left\{ \begin{aligned} \frac{\sigma^2}{\sigma_0^2} = \frac{(m-1)\, \varphi^2 + (n-1)\, \gamma^2 + p(1-p)}{\frac{1}{12}\, (m+n+1)} \leqq \frac{12\, (m+n-1)}{(m+n+1)}\, p(1-p) < \\ < 12\, p(1-p) \leqq 3 \end{aligned} \right.$$ [4] [5]

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[4]) By means of the

Lemma. If, for $0 \leqq x \leqq 1$, $1°$ $x + f(x)$ *is monotonous non-decreasing,* $2°$ $\int_0^1 f(x)\, \mathrm{d}x = 0$, $3°$ $f(1) \leqq f(0)$, *then* $\int_0^1 f(x)^2 \mathrm{d}x \leqq \frac{1}{12}$,

it follows even that $\frac{\sigma^2}{\sigma_0^2} \leqq 12\, p(1-p)$. $\mathrm{Max}\,(m, n)/(m+n+1)$. The equality sign holds for continuous F and G if and only if either $m \leqq n$, $\mathrm{P}[\mathbf{y} \leqq \text{all } \mathbf{x}] = p$ and $\mathrm{P}[\mathbf{y} \geqq \text{all } \mathbf{x}] = 1 - p$ (i.e. $\gamma^2 = 0$, $\varphi^2 = p(1-p)$),
or $n \leqq m$, $\mathrm{P}[\mathbf{x} \leqq \text{all } \mathbf{y}] = 1 - p$ and $\mathrm{P}[\mathbf{x} \geqq \text{all } \mathbf{y}] = p$ (i.e. $\varphi^2 = 0$, $\gamma^2 = p(1-p)$).
Hence σ^2/σ^2_0 can come arbitrarily near to its upper bound 3, if and only if $p = \frac{1}{2}$ and either m/n or n/m tends to zero. If $m = n$ $\sigma^2/\sigma^2_0 < 6\, p(1-p) \leqq \frac{3}{2}$.

[5]) Although the relation is not needed in this paper, we may remark that also a lower boundary can be found for the ratio σ^2/σ^2_0 by means of the inequalities $\varrho^2 \leqq 1$ for the correlation coefficients between $F(\mathbf{x})$ and $G(\mathbf{x})$ and between $F(\mathbf{y})$ and $G(\mathbf{y})$. Introducing the abbreviation $A = 4\, p(1-p)$, so that $0 \leqq A \leqq 1$, we find that

$$12\, \varphi^2 \geqq (\sqrt{3A+1} - 1)^2, \quad 12\, \gamma^2 \geqq (\sqrt{3A+1} - 1)^2,$$

and, remarking that $A \geqq \sqrt{(3A+1} - 1)^2$ for $0 \leqq A \leqq 1$,

$$\frac{\sigma^2}{\sigma_0^2} \geqq (\sqrt{3A+1} - 1)^2 + 3\, \frac{A - (\sqrt{3A+1} - 1)^2}{m+n+1} \geqq (\sqrt{3A+1} - 1)^2.$$

The last member vanishes for $p = 0$ and for $p = 1$, i.e. if all $\mathbf{x}$ are $\leqq$ all $\mathbf{y}$ or

The hypothesis H_0 is rejected if and only if $\boldsymbol{U} \leqq \mu_0 - c\sigma_0$.

Hence the probability under the hypothesis H that H_0 is not rejected is

$$\text{(II)} \left\{\begin{aligned} & \mathrm{P}[\boldsymbol{U} > U_{a,m,n} \mid H] = P[\boldsymbol{U} > \mu_0 - c\sigma_0 \mid H] = \\ &= \mathrm{P}[\boldsymbol{U} - \mu > (\tfrac{1}{2} - p)\, mn - c\sigma_0 \mid H] \leqq \frac{\sigma^2}{\{(\frac{1}{2} - p)\, mn - c\sigma_0\}^2} = \\ &\qquad = \frac{\sigma^2}{\sigma_0^2}\left\{(\tfrac{1}{2} - p)\left(\frac{12\, mn}{m+n+1}\right)^{\frac{1}{2}} - c\right\}^{-2} \end{aligned}\right.$$

because of BIENAYMÉ's inequality, $(\frac{1}{2} - p)\, mn - c\sigma_0$ being *positive* for sufficiently large $mn/(m + n)$ as c is bounded and $p < \frac{1}{2}$.

In the last member the first factor $\sigma^2/\sigma_0{}^2$ is bounded, as we saw above. The expression between curved brackets tends to infinity with $mn/(m + n)$, c being bounded and p being $< \frac{1}{2}$.

Hence the whole last member tends to zero, and

$$\lim \mathrm{P}\, [\boldsymbol{U} \leqq U_{a,m,n} \mid H] = 1,$$

which proves the first part of the theorem.

3. We now prove that the class of alternatives cannot be extended without loss of the property of consistency. We assume therefore $\mathrm{P}[\boldsymbol{y} < \boldsymbol{x}] = p \geqq \frac{1}{2}$, F and G being continuous as before.

The probability of rejection is now

$$\text{(III)} \left\{\begin{aligned} & \mathrm{P}\,[\boldsymbol{U} \leqq \mu_0 - c\sigma_0 \mid H] = \mathrm{P}\,[\boldsymbol{U} - \mu \leqq -\{(p - \tfrac{1}{2})\, mn + c\sigma_0\} \mid H] \leqq \\ & \qquad \leqq \frac{\sigma^2}{\{(p - \frac{1}{2})\, mn + c\sigma_0\}^2} = \frac{\sigma^2}{\sigma_0^2}\left\{(p - \tfrac{1}{2})\left(\frac{12\, mn}{m+n+1}\right)^{\frac{1}{2}} + c\right\}^{-2} \end{aligned}\right.$$

because, again, of BIENAYMÉ's inequality, and of the fact that $(p - \frac{1}{2})\, mn + c\sigma_0$ is now *positive* for all $p \geqq \frac{1}{2}$ [6]) and all sufficiently small α (for all but the smallest values of m and n $\alpha < \frac{1}{2}$ will be sufficient).

For $p > \frac{1}{2}$ the last member tends to zero, as before, so that in that case the hypothesis H_0, although it implies $p = \frac{1}{2}$, will *not* be rejected with a probability tending to 1. It is clear that this is due to the use of a onesided (leftsided) test. This part of our theorem may also be derived directly from inequality II, from which it follows by symmetry. The inequality II, however, is sharper than the one obtained in this way.

For $p = \frac{1}{2}$ the last member is $\leqq \sigma^2/\sigma_0^2 c^2 \leqq \frac{3}{c^2}$. Hence as soon as α is sufficiently small, so that $c > \sqrt{3}$, there remains a *positive* probability $\geqq 1 - 3/c^2$ that the hypothesis H_0 will not be rejected. This probability

reversely. In that case $\boldsymbol{U}$ can take on only the value 0 or $m\, n$ respectively, whence $\sigma = 0$, so that in these two cases and only then the inequality becomes an equality. As for $0 \leqq A \leqq 1$ $\sqrt{3A + 1} - 1 \geqq A$, we have, somewhat more roughly,

$$16\, p^2 (1 - p)^2 \leqq \frac{\sigma^2}{\sigma_0^2} < 12\, p\, (1 - p).$$

[6]) Even for $p > \frac{1}{2} - c\, \sigma_0/mn = \frac{1}{2} - c \sqrt{\dfrac{m+n+1}{12\, mn}}$.

tends to 1 if $\alpha \to 0$. As $\sqrt{3} \approx 1{,}732 \approx \xi_{0.0415}$, $\alpha \leqq 0{,}04$ will in any case be sufficient [7]). This estimate, however, is far too small because of the roughness of BIENAYMÉ's inequality. If the distribution of $\boldsymbol{U}$ under $\mathcal{H}$ also is near to normality [8]), the value

$$1 - \mathrm{P}\,[\mu - \boldsymbol{U} \geqq c\,\sigma_0 \,|\, \mathcal{H}] \approx \phi(c\,\sigma_0/\sigma) \geqq \phi(\xi_\alpha/\sqrt{3})$$

where ϕ denotes the normal cumulative distribution function, will yield a better estimate for the limit probability of nonrejection, and shows in any case that it is $>\frac{1}{2}$ for whichever $\alpha<\frac{1}{2}$, provided that $\boldsymbol{U}$ is sufficiently nearly normally distributed. It must, however, be remarked that $\boldsymbol{U}$ is not known to be asymtotically normal under hypotheses different from $\mathcal{H}_0$ [8]).

4. Because of the well-known difficulty of the problem, to determine exactly the power function of WILCOXON's test, it may be of some use, to draw attention to some partial results, recently obtained in this direction by H. R. VAN DER VAART. He considered the case only where $F(x)$ and $G(x)$ are normal with unit variance and difference of means $=\mu$. If $\alpha_+(\mu)$ and $\alpha_\pm(\mu)$ denote the power functions of the leftsided and the twosided tests respectively, he succeeded in his far from easy investigation to express the first and second derivatives $\alpha'_+\,(0)$ and $\alpha''_\pm(0)$ respectively by means of the (spherical) volumes of spherical simplices in ν dimensions, where $\nu = m + n - 3$ and $\nu = m + n - 4$ respectively. As the purely mathematical problem of determining this volume is unsolved for $\nu \geqq 3$, he could compute $\alpha'_+(0)$ and $\alpha''_\pm(0)$ only for $m + n \leqq 5$ and $m + n \leqq 6$ respectively. He also discussed the conditions under which the tests are unbiased and showed them by counterexamples to be relevant.

As there seems to be a rather widespread opinion that rank invariant methods have a low efficiency in comparison with the parametric ones (in the cases where the latter are applicable), it is noteworthy that the differences in $\alpha'_+(0)$ and $\alpha''_\pm(0)$ between STUDENT's and WILCOXON's test are relatively small.

The computed differences are less than $2\frac{1}{2}$ % of their values in the case of the onesided and less than about 6 % in the case of the twosided test. That this does not hold for these particular values of m and n alone follows from another result of VAN DER VAART, according to which the limit of

$$\alpha''_\pm(0)_{\text{Wilc.}}/a''_\pm(0)_{\text{Stud.}} \text{ equals } 3/\pi.$$

Moreover we might remark that, strictly speaking, one should compare

7) If $m = n$, $\sigma^2/\sigma^2_0 \leqq \frac{3}{2}$ (cf. [4])), so that $c > \sqrt{3/2}$ is already sufficient leading to $\alpha \lesssim 0{,}11$.

8) Since the paper was completed mr. A. M. MOOD kindly informed me that Mr. E. LEHMANN has proved the important result that the WILCOXON test criterion is asymptotically normally distributed even when the null hypothesis is not true.

WILCOXON's test for given m, n, and α with STUDENT's test not for the same values m and n, but for values $m + m'$, $n + n'$, $\alpha + \alpha'$ where m' and n' observations respectively are used to ascertain on a level of significance α' the applicability of STUDENT's test, i.e. the normality of the two distributions and the equality of the variances. Together with the fact that, at least for small values of m and n, WILCOXON's test requires far less computational work than STUDENT's, this has lead us, guided originally by MANN and WHITNEY's excellent paper, to make a rather extensive use of WILCOXON's test at the Statistical Department of the Mathematical Centre at Amsterdam, and with very satisfactory results, showing by experience also that the efficiency of this test is quite sufficient for most practical purposes.

5. As a further contribution towards the determination of the power function for WILCOXON's test, it may be remarked that the inequalities, I II, III allow the determination of a lower boundary for this quantity. In fact, α_{H} being the value of the power function for the hypothesis H, I and II lead to

$$\text{(IV)} \qquad \alpha_{H} \geqq 1 - 3\left\{\left(\tfrac{1}{2} - p\right)\left(\frac{12mn}{m+n+1}\right)^{\frac{1}{2}} - c_\beta\right\}^{-2}, \text{ [9])}$$

provided the expression between curved brackets is positive. In order that

$$\alpha_{H} \geqq 1 - \alpha^*$$

it is therefore sufficient that

$$\text{(V)} \qquad p \geqq \tfrac{1}{2} - \left(\frac{m+n+1}{12mn}\right)^{\frac{1}{2}}\left(c_\beta + \sqrt{\frac{3}{\alpha^*}}\right),$$

or, equivalent with this inequality, if $p<\frac{1}{2}$,

$$\frac{12mn}{m+n+1} \geqq \frac{1}{(\frac{1}{2}-p)^2}\left(c_\beta + \sqrt{\frac{3}{\alpha^*}}\right)^2.$$

As soon as these relations are satisfied, the hypothesis H_0, will be rejected on a true level of significance β, except for a probability $\leqq \alpha^*$.

On the other hand I and III show that

$$\text{(VI)} \qquad \alpha_{H} \leqq 3\left\{\left(p - \tfrac{1}{2}\right)\left(\frac{12mn}{m+n+1}\right)^{\frac{1}{2}} + c_\beta\right\}^{-2},$$

provided the expression between curved brackets is positive, i.e. if

$$p \geqq \tfrac{1}{2} - \left(\frac{m+n+1}{12mn}\right)^{\frac{1}{2}} c_\beta.$$

In this case we have

$$\alpha_{H} \leqq \alpha^{**}$$

if

$$\text{(VII)} \qquad p \geqq \tfrac{1}{2} - \left(\frac{m+n+1}{12mn}\right)^{\frac{1}{2}}\left(c_\beta - \sqrt{\frac{3}{\alpha^{**}}}\right),$$

[9]) c_β is the coefficient $c = (U - \mu_0)/\sigma_0$ corresponding to a true level of significance β.

which, for $p \geqq \frac{1}{2}$ and $\alpha^{**} \geqq 3c_\beta^{-2}$ is always satisfied, and for $p > \frac{1}{2}$ and $\alpha^{**} < 3c_\beta^{-2}$ is equivalent with

$$\frac{12\,mn}{m+n+1} \geqq \frac{1}{(p-\frac{1}{2})^2}\left(\sqrt{\frac{3}{\alpha_{**}}} - c_\beta\right)^2.$$

For $p \leqq \frac{1}{2}$ it can hold only with $\alpha^{**} > 3c_\beta^{-2}$ and then, if $p < \frac{1}{2}$ is equivalent with

$$\frac{12\,mn}{m+n+1} \leqq \frac{1}{(\frac{1}{2}-p)^2}\left(c_\beta - \sqrt{\frac{3}{\alpha^{**}}}\right)^2.$$

6. Resuming our results we can state:

1. The one-sided WILCOXON-test is consistent against the class of all (cf. 2)) hypotheses for which $p = \mathrm{P}\,[\mathbf{y} < \mathbf{x}] < \frac{1}{2}$ and, for sufficiently small α, against no other ones.
2. WILCOXON's two-sided test is consistent against the class of all hypotheses for which $p \neq \frac{1}{2}$, and, for sufficiently small α, against no other ones.
3. The hypothesis H_0, when tested on a true level of significance β, will be rejected spr α^* [10]) if V is satisfied, i.e. if $p < \frac{1}{2}$ and if, ceteris paribus, p [11]) is sufficiently small, or $\frac{mn}{m+n}$ or β [12]) or α^* [13]) sufficiently large.
4. Testing the hypothesis H_0 on a true level of significance β leads spr α^{**} to non-rejection if VII is satisfied, i.e. if either
   1°. $p > \frac{1}{2}$, and if, ceteris paribus $p - \frac{1}{2}$ [14]) or $\frac{mn}{m+n}$ or α^{**} [15]) or β [16]) is sufficiently large,
   2°. $p < \frac{1}{2}$ and $\alpha^{**} > 3c_\beta^{-2}$, and if, ceteris paribus, $\frac{1}{2} - p$ or

[10]) It has been found useful to introduce the abbreviations 'spr p', to read "salva probabilitate p", as an abbreviation for the expression "except for a probability $\leqq p$". Hence, A being a statement about random variables, the formula A spr p is defined as $\mathrm{P}[\mathbf{A}] \geqq 1 - p$.

[11]) Provided $\sqrt{\frac{3\,mn}{m+n+1}} \geqq c_\beta + \sqrt{\frac{3}{\alpha^*}}$.

[12]) Provided $(\frac{1}{2}-p)\sqrt{\frac{12\,mn}{m+n+1}} \geqq \sqrt{\frac{3}{\alpha^*}}$.

[13]) Provided $(\frac{1}{2}-p)\sqrt{\frac{12\,mn}{m+n+1}} \geqq c_\beta + \sqrt{3}$.

[14]) Provided $\sqrt{\frac{3\,mn}{m+n+1}} \geqq \sqrt{\frac{3}{\alpha^{**}}} - c_\beta$.

[15]) Provided $(p-\frac{1}{2})\sqrt{\frac{12\,mn}{m+n+1}} \geqq \sqrt{3} - c_\beta$.

[16]) Sufficient is $c_\beta \geqq \sqrt{\frac{3}{\alpha^{**}}}$.

$\frac{mn}{m+n}$ or β is sufficiently small or α^{**} [17]) sufficiently large, or

3°. $\varrho = \frac{1}{2}$ and $\alpha^{**} \geqq 3c_\beta^{-2}$.

5. For $p = \frac{1}{2}$ non-rejection spr α^{**} can not necessarily be obtained by augmenting m and n, but only by decreasing β hence by testing H_0 sufficiently sharply.

6. Results analogous with those of 3, 4.2°, 4.3° and 5, hold for the two-sided test with regard to $p \neq \frac{1}{2}$ and $p = \frac{1}{2}$.

7. Apart from the restriction, made in 5, WILCOXON's test can be considered as a *test for the median of* $\mathbf{x} - \mathbf{y}$.

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[17]) Provided $0 < (\frac{1}{2} - p)\sqrt{\frac{12\,mn}{m+n+1}} \leqq c_\beta - \sqrt{3}$.